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Diffraction of water waves by a submerged vertical plate

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A thin vertical plate makes small, simple harmonic rolling oscillations beneath the surface of an incompressible, irrotational liquid. The plate is assumed to be so wide that the resulting equations may be regarded as two-dimensional. In addition, a train of plane waves of frequency equal to the frequency of oscillation of the plate, is normally incident on the plate. The resulting linearized boundary-value problem is solved in closed form for the velocity potential everywhere in the fluid and on the plate. Expressions are derived for the first- and second-order forces and moments on the plate, and for the wave amplitudes at a large distance either side of the plate. Numerical results are obtained for the case of the plate held fixed in an incident wave-train. It is shown how these results, in the special case when the plate intersects the free surface, agree, with one exception, with results obtained by Ursell (1947) and Haskind (1959) for this problem.

1. Introduction

The linearized equations describing the generation or scattering of water waves by an obstacle do not, in general, permit a closed form solution. There are a few exceptions, however, the most notable being when the obstacle is a vertical plate or plates and the motion may be regarded as two-dimensional. Thus, Dean (1945) first solved the problem of diffraction of waves by a submerged semi-infinite vertical barrier which extended to a finite distance beneath the surface. Shortly afterwards, Ursell (1947, 1948) considered the diffraction of waves by a finite vertical barrier which intersected the free surface, and also the waves produced by the rolling motion of such a barrier. A number of people took up various aspects of this problem at a later date, including Haskind (1948, 1959), who derived expressions in terms of Bessel and related functions for the forces and moments on a barrier oscillating in an arbitrary manner in a train of incident waves of the same frequency. Two authors, Lewin (1963) and Mei (1966), have considered the very general problem of the generation and scattering of waves by an arbitrary number of vertical plates; in each case the author considers as a special case the surface-piercing barrier of Ursell.

In this paper, the problem to be solved is the generation and diffraction of waves by a submerged finite vertical plate. This may also be regarded as a special case of the theory of Lewin or Mei, but for completeness the theory is derived afresh. The plate makes small rolling oscillations in a train of incident waves of the same frequency as the plate oscillations. Expressions are obtained for the wave amplitude at a distance from the plate, and also for the first- and second-order forces and moments acting on the plate. The expressions involve combinations of various integrals which, in special limiting cases, become either Bessel functions or elliptic functions.

The problem of the oscillating submerged plate in a stationary fluid has been discussed by Wehausen (1960, p. 560 *et seq.*), using a different method from that employed here. He assumes that the vertical velocity is zero near the edges of the plate and derives the solution by solving in closed form a pair of integral equations of airfoil type. He then needs to specify the circulation around the plate in order to completely determine the solution. The details of the solution are omitted.

Experiments carried out by Keulegan & Carpenter (1958) indicate that the parameter $\pi l/D$ is fundamental in determining the amount of vorticity shed at the edges of a submerged body in waves. Here, D is a typical length of the body normal to the direction of the waves, and l is the distance a fluid particle travels during a half cycle in the absence of the body. Thus the smaller the value of $\pi l/D$, the less significant is the eddy formation at the edges of the body. Now the fluid particles in deep water describe circles whose maximum radius (at the water surface) is equal to the wave amplitude (Lamb 1932, p. 368) so that the assumption made in this paper, namely that the flow is irrotational and no vortex shedding occurs at the edges of the plate, is not unreasonable in terms of a theory based on waves of infinitesimal amplitude.

The results simplify considerably if the plate is assumed to be held fixed in a train of incident waves, and computations are limited to this case. Of particular interest is the existence of a mean second-order vertical force acting on the plate, although the plate has no thickness. This arises from the singularity in vertical velocity at either end of the plate, which, from Bernoulli's equation, produces an unbounded negative pressure acting at the ends. By a suitable limiting procedure (Sedov 1965, p. 52), this may be shown to yield a finite vertical force at each end, the resultant of which is directed towards the free surface. A similar suction force is exerted on a flat plate in aerodynamic theory. See, for example, Robinson & Laurmann (1956, p. 126).

The only other case, in which the forces on a submerged body in waves have been determined on the basis of the full linearized theory, seems to be the work of Ogilvie (1963). Thus Ursell (1950) reduced the linearized potential problem of water waves passing over a submerged circular cylinder to the solution of an infinite system of equations which were well suited to computation. Ogilvie computed the first-order oscillatory force for the fixed cylinder and also extended Ursell's work to consider a cylinder oscillating sinusoidally in otherwise calm water, and a neutrally buoyant cylinder which is allowed to respond to the first-order oscillatory forces. He showed that knowledge of the first-order potential was sufficient to determine the mean second-order forces on the cylinder, which he then computed. Similar results are obtained in the present work.

The paper is divided into sections. The formulation and solution of the mathematical problem of determining the velocity potential of the flow is dealt with in §§ 2–4. In § 5 general expressions are presented for the forces and moments on the oscillating plate in waves, whilst in § 6 the plate is assumed to be held fixed in waves and the simplified results for the forces and moments are derived. The limiting case when the upper edge of the plate approaches the free surface is considered in § 7, and a comparison is made between the limiting values of the various expressions and the values obtained by Ursell (1947) and Haskind (1948, 1959) for the diffraction of waves by a vertical surface-piercing plate. The essential results of the paper are given in § 8, where figures 2–7, illustrating the various physical expressions derived in § 6 for the diffraction of waves by a fixed plate, are discussed.

It should be noted that the solution to the fixed submerged plate was outlined by Ursell (1947) as a straightforward extension of the surface-piercing barrier problem, although the details were omitted.

2. Formulation

Cartesian co-ordinates are chosen with y directed vertically upwards and the origin in the free surface. The plate occupies the interval $L\colon x = 0,\ -b < y < -a$, and makes small simple harmonic rolling motions of amplitude θ_0 and frequency σ about a fixed point $-c$ which need not be in L. The case of sway is achieved by letting $c \to \infty$ and $\theta_0 \to 0$ such that $c\theta_0$ remains finite. There is also a train of waves incident from $x = +\infty$, of amplitude a_0 and frequency σ.

The resulting fluid motion will also have frequency σ, and it is convenient to write the total wave potential $\Phi(x, y, t)$ as

$$\Phi(x,y,t) = \mathrm{Re}_j\{\phi(x,y)\, e^{-j\sigma t}\} \tag{1}$$

and the incident wave potential as

$$\Phi_0(x,y,t) = \mathrm{Re}_j\{\phi_0(x,y)\, e^{-j\sigma t}\}, \tag{2}$$

where $j = \sqrt{-1}$, and so that

$$\phi_0(x,y) = \frac{-jga_0}{\sigma}\, e^{Ky-jKx}, \quad K = \sigma^2/g, \tag{3}$$

with incident amplitude obtained from

$$\eta_0(x,t) = -\frac{1}{g}\frac{\partial \Phi_0}{\partial t} = a_0 \cos(Kx+\sigma t). \tag{4}$$

The use of the complex number j serves to simplify the time-dependence of the velocity potential. Later on the complex variable $z = x+iy$, where $i = \sqrt{-1}$ will be introduced. This use of two complex numbers need not cause confusion, provided the distinct roles of i and j are kept in mind.

To first order in the amplitude, the velocity of the plate may be written

$$\frac{\partial \Phi}{\partial x} = \sigma\theta_0(c+y)\cos\sigma t \quad \text{on } L, \tag{5}$$

so that, from (1),

$$\frac{\partial \phi}{\partial x} = \sigma\theta_0(c+y) \quad \text{on } L. \tag{6}$$

In addition, the usual assumptions of the linearized theory of water waves (Wehausen & Laitone 1960) provide the following equations for $\phi(x,y)$:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{in the fluid,} \tag{7}$$

$$K\phi-\frac{\partial\phi}{\partial y} = 0 \quad y = 0, \quad -\infty < x < \infty, \quad K = \sigma^2/g. \tag{8}$$

From the form of (1) and (3), a solution $\phi(x,y)$ is required which behaves in the following way for large x:

As $x \to +\infty$,
$$\phi(x,y) \sim A_+ e^{Ky+jKx} - (jga_0/\sigma)\, e^{Ky-jKx}, \tag{9}$$
while, as $x \to -\infty$,
$$\phi(x,y) \sim A_- e^{Ky-jKx} \tag{10}$$
for some constants $A_\pm$.

Finally, the velocity components are assumed to be bounded everywhere and tend to zero as $y \to -\infty$, except at the edges of the plate where a mild singularity is allowed.

Thus it is assumed that
$$\frac{\partial\phi}{\partial r} = O\left(\frac{1}{r^\beta}\right), \quad 0 < \beta < 1 \tag{11}$$
in the neighbourhood of the point $(0, -a)$ or $(0, -b)$. Here, r is the distance from a point in the fluid to either of these points.

3. Method of solution

We introduce the complex potential $w(z) = \phi + i\psi$, $z = x+iy$, $i = \sqrt{-1}$, where $\psi(x,y)$ is the two-dimensional stream function. Then it is convenient to consider the conditions satisfied by the so-called reduced potential $W(z)$ defined by the equation,
$$W(z) = \frac{dw}{dz}+iKw. \tag{12}$$

Thus, it is easily verified from condition (8) that
$$\mathrm{Im}_i\, W(z) = 0, \quad z \text{ real.} \tag{13}$$

Hence $W(z)$ may be continued by Schwarz's reflexion principle into $y > 0$, where
$$W(\bar{z}) = \overline{W(z)}. \tag{14}$$

Furthermore, $W(z)$ is a single-valued function outside the circle $z = b$, and has a Laurent expansion of the form
$$W(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$$

It is assumed that (9) and (10) may be differentiated once with respect to x or y. Then it follows that $W(z) = O(1)$ for $z \to \pm\infty$. But $dw/dz = o(1)$ as $|z| \to \infty$, $\mathrm{Im}\, z < 0$, and hence, using (12) and (14), $W(z) = o(z)$ as $|z| \to \infty$ for all z, so that $c_n \equiv 0$, $n > 0$, and hence
$$W(z) = O(1) \tag{15}$$
in a full neighbourhood of infinity.

Now since $\partial\phi/\partial x = \partial\psi/\partial y$, we have by integration,

$$\psi(0,y) = \theta_0(cy+\tfrac{1}{2}y^2) \quad \text{on } L, \tag{16}$$

where ψ has been defined so that the constant of integration is zero. But

$$\mathrm{Re}_i\, W(z) = \frac{\partial\phi}{\partial x} - K\psi, \tag{17}$$

so that from (6) and (16)

$$\mathrm{Re}_i\, W(z) = f(y) \quad \text{on } L,$$

where

$$f(y) = \sigma\theta_0(c+y(1-Kc)-\tfrac{1}{2}Ky^2). \tag{18}$$

Equation (17) is the key to the solution of the problem. Because the plate is vertical, it is possible to express the boundary condition on the plate in terms of the reduced potential $W(z)$, and the problem reduces to the determination of $W(z)$ and then the solution of the simple differential equation (12) for $W(z)$. By introducing a more complicated reduced potential, it is possible to solve a variety of problems by this method. For example, John (1948) has applied the method to solve, in principle, the problem of the diffraction of water waves by a surface-piercing barrier inclined at an angle $\pi/2n$ (n integer) to the horizontal.

Note that from (14), we have

$$\mathrm{Re}_i\, W(z) = f(-|y|) \quad \text{on } L+L', \tag{19}$$

where L' is the interval $x = 0$, $a < y < b$, the reflexion of L in the real axis.

Finally, $W(z)$ may be unbounded near the ends of L, and by reflexion, the ends of L'. Thus,

$$W(z) = O\left(\frac{1}{(z^2+a^2)^\beta}\right), \quad 0<\beta<1, \quad \text{near } z = \pm ia, \tag{20}$$

with a similar behaviour near $z = \pm ib$.

The problem of determining the most general function $W(z)$ satisfying (15), (19) and (20) is a typical Riemann–Hilbert problem, and the procedure is well established. Only the final result will be given; an extensive treatment of such problems can be found in Muskhelishvili (1963, p. 261 *et seq.*). The solution is not unique, but depends upon the assumed behaviour near the ends of L and L'. We shall choose a solution which is singular at the ends of L and L'. It turns out that this is the only choice which allows us to satisfy all the conditions of the problem for $W(z)$.

Thus it may be shown that

$$W(z) = \frac{1}{((z^2+a^2)(z^2+b^2))^{\frac{1}{2}}}\left\{B+Cz^2+\frac{2}{\pi}\int_{-b}^{-a}\frac{((\eta^2-a^2)(b^2-\eta^2))^{\frac{1}{2}}\,\eta f(\eta)}{\eta^2+z^2}\,d\eta\right\}, \tag{21}$$

with B, C real constants with respect to i, satisfies (15), (19) and (20).

It is noteworthy that alternative, but equivalent, forms are possible for $W(z)$. Thus the square-root factors in the expression for $W(z)$ may appear as ratios without affecting the general expression for $W(z)$ (Muskhelishvili 1963, p. 237).

4. The solution for the complex potential $w(z)$

The potential $w(z)$ is obtained from the reduced potential $W(z)$ by integration of (12). Thus,

$$w(z) = e^{-iKz}\left\{A + \int_{-ia}^{z} e^{iKu} W(u)\,du\right\}, \tag{22}$$

where A is an arbitrary real constant with respect to i.

It may be verified that the form of (22) is such that condition (16) is satisfied.

It remains to determine the constant A, and the constants B and C in the definition of $W(z)$.

Now the value of the complex velocity potential at a point z $(= x+iy)$ in the fluid is given by (22). It will be assumed, as mentioned in the introduction, that the circulation around the plate is zero, which implies that the potential is single-valued within the fluid. To satisfy this condition, it follows from (22) that

$$\mathrm{Re}_i \oint_C e^{iKu} W(u)\,du = 0,$$

where C is a closed contour surrounding L.

If the path of integration is contracted onto L, this condition may be written

$$\int_{-b}^{-a} \frac{e^{-Ku}}{((u^2-a^2)(b^2-u^2))^{\frac{1}{2}}}\left\{B - Cu^2 + \frac{2}{\pi}\int_{-b}^{-a} \frac{((\eta^2-a^2)(b^2-\eta^2))^{\frac{1}{2}}\,\eta f(\eta)}{\eta^2-u^2}\,d\eta\right\}du = 0 \tag{23}$$

which determines B in terms of C.

The remaining unknowns are determined by imposing conditions (9) and (10) on the solution (22). To do this, we required the behaviour of the integral in (22) as $|z| \to \infty$ in $y \leqslant 0$.

Now $W(z)$ is bounded at infinity and $W(z) - C \to 0$ as $|z| \to \infty$. So

$$\int_{-ia}^{z} e^{iKu} W(u)\,du = \int_{-ia}^{i\infty} e^{iKu} W(u)\,du + \int_{i\infty}^{z} e^{iKu}\{W(u) - C\}\,du - \frac{iC}{K} e^{iKz}, \tag{24}$$

and the path of integration in the first integral on the right-hand side is indicated by the vertical dotted line in figure 1. As $z \to +\infty$ along the path C_+, the second integral on the right-hand side tends to zero by Jordan's lemma, since

$$W(z) - C \to 0 \quad \text{as} \quad |z| \to \infty.$$

If the path of integration along C_- to $z = -\infty$ is treated similarly, then the values obtained for the integral in (22) differ by a contour integral around L'. If this is contracted onto L', it follows that, for $z \to \pm\infty$,

$$w(z) \sim e^{-iKz}\{A \mp \gamma + i(\alpha - \beta + \delta)\} - (iC/K), \tag{25}$$

where

$$\gamma = Ba_1(K) - Ca_1''(K) + a_1(K, F), \tag{26}$$

$$\alpha = Ba_2(K) - Ca_2''(K) + a_2(K, F), \tag{27}$$

$$\beta = Ba_3(K) - Ca_2''(K) + a_3(K, F), \tag{28}$$

and
$$a_1(K,F) = \int_a^b \frac{e^{-Ku}F(a,b,u)}{((u^2-a^2)(b^2-u^2))^{\frac{1}{2}}}\,du, \tag{29}$$

$$a_2(K,F) = \int_{-a}^{a} \frac{e^{-Ku}F(a,b,u)}{((a^2-u^2)(b^2-u^2))^{\frac{1}{2}}}\,du, \tag{30}$$

$$a_3(K,F) = \int_b^{\infty} \frac{e^{-Ku}F(a,b,u)}{((u^2-a^2)(u^2-b^2))^{\frac{1}{2}}}\,du, \tag{31}$$

$$F(a,b,u) = \frac{2}{\pi}\int_{-b}^{-a} \frac{((\eta^2-a^2)(b^2-\eta^2)\,\eta)^{\frac{1}{2}}f(\eta)}{\eta^2-u^2}\,d\eta, \tag{32}$$

$$\delta(K) = \int_a^b e^{-Ku}f(-u)\,du, \tag{33}$$

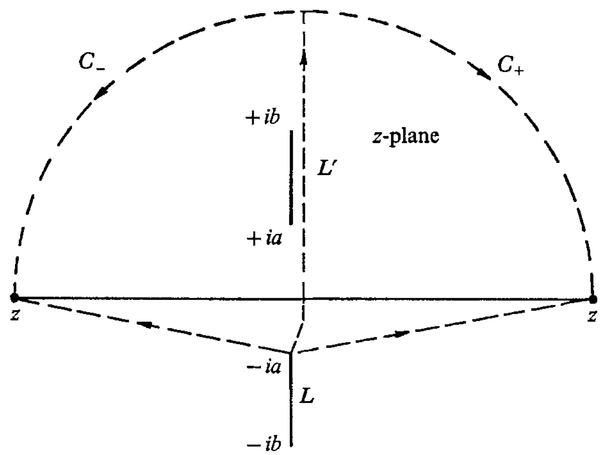

FIGURE 1. Path of integration for $w(z)$ as $z \to \pm\infty$.

and the notation

$$a_i(K,1) \equiv a_i(K), \quad a_i''(K) \equiv \frac{d^2a_i(K)}{dK^2} \quad (i = 1,2,3)$$

is used.

The expression for $F(a,b,u)$ from (32) with $f(y)$ given by (18) may be expressed as combinations of complete elliptic integrals, but there would appear to be little advantage in doing this.

Note that in the above notation, (32) may be written

$$\gamma(-K) = Ba_1(-K) - Ca_1''(-K) + a_1(-K,F) = 0. \tag{34}$$

Now from (25), as $x \to \pm\infty$,

$$\phi(x,y) \sim \{(A \mp \gamma)\cos Kx + (\alpha-\beta+\delta)\sin Kx\}\,e^{Ky}. \tag{35}$$

It follows, by comparison of (35) with the desired behaviour as $x \to \pm\infty$ given by (9) and (10), that

$$A = -jga_0/\sigma \tag{36}$$

and
$$\gamma = j(\alpha-\beta+\delta+ga_0/\sigma), \tag{37}$$

so that, as $x \to +\infty$,

$$\phi(x,y) \sim -\gamma(K)\, e^{jKx+Ky} - (jga_0/\sigma)\, e^{-jKx+Ky}, \tag{38}$$

and, as $x \to -\infty$,
$$\phi(x,y) \sim (\gamma(K) - jga_0/\sigma)\, e^{-jKx+Ky}, \tag{39}$$

and the solution is uniquely determined.

Thus, after some algebra, it may be shown that

$$C = \frac{-\Delta_{21} + j\Delta_3}{\Delta_{11}(1+j\tau)}, \tag{40}$$

where
$$\Delta_{1i} = \begin{vmatrix} a_i(K) & a_1(-K) \\ a_i''(K) & a_1''(-K) \end{vmatrix}, \tag{41}$$

$$\Delta_{2i} = \begin{vmatrix} a_i(K) & a_1(-K) \\ a_i(K,F) & a_1(-K,F) \end{vmatrix} \quad (i = 1, 2), \tag{42}$$

$$\Delta_3 = \Delta_{22} - \Delta_{23} - a_1(-K)\{ga_0/\sigma + \delta(K)\}, \tag{43}$$

and
$$\tau = \frac{\Delta_{12} - \Delta_{13}}{\Delta_{11}}. \tag{44}$$

It follows from (34), (37) and (40) that

$$\gamma(K) = \gamma_1 + j\gamma_2 = -\frac{(\Delta_2 + \tau\Delta_{21})}{a_1(-K)} \frac{1}{\tau - j}. \tag{45}$$

Finally,
$$\Phi(x,y,t) = \mathrm{Re}_j\{\phi(x,y)\, e^{-j\sigma t}\}, \tag{46}$$

where
$$\phi(x,y) = \mathrm{Re}_i\{w(z)\},$$

and $w(z)$ is given by (22) with $W(z)$ given by (21), and the constants A, B and C, by (34), (36) and (40).

The velocity potential on the plate is

$$\mathrm{Re}_j\{\phi(0,y)\, e^{-j\sigma t}\},$$

where
$$\phi^{\pm}(0,y) = \frac{-jga_0}{\sigma} e^{Ky} \pm e^{Ky}\gamma(-K,y), \quad y \text{ on } L, \tag{47}$$

and
$$\gamma(-K,y) = \int_{-b}^{y} \frac{e^{-Ku}}{((u^2-a^2)(b^2-u^2))^{\frac{1}{2}}} \{B - Cu^2 + F(a,b,u)\}\, du. \tag{48}$$

Note that $\gamma(-K,-b) = 0$ and $\gamma(-K,-a) \equiv \gamma(-K) = 0$ from condition (34).

5. The forces and moments acting on the plate

The first-order force and moment acting on the plate are obtained by integrating the linearized expression for the pressure around the boundary of the plate.

We have

$$\begin{aligned} p(x,y,t) &= -\rho \frac{\partial \Phi}{\partial t}(x,y,t) - \rho g y \\ &= \mathrm{Re}_j\{\rho\sigma j\phi\, e^{-j\sigma t}\} - \rho g y, \end{aligned} \tag{49}$$

so that the force per unit width of the plate is

$$X^{(1)}(t) = \int_{-b}^{-a} \{p^-(0,y,t) - p^+(0,y,t)\}\,dy, \tag{50}$$

$$= -\mathrm{Re}_j \left\{ 2\rho\sigma j\, e^{-j\sigma t} \int_{-b}^{-a} e^{Ky}\, \gamma(-K,y)\,dy \right\}, \tag{51}$$

from (47)

$$= \mathrm{Re}_j \frac{2\rho\sigma j\, e^{-j\sigma t}}{K} \int_{-b}^{-a} \frac{1}{((y^2-a^2)(b^2-y^2))^{\frac{1}{2}}} \{B - Cy^2 + F(a,b,y)\}\,dy \tag{52}$$

after integration by parts.

Similarly, the first-order moment per unit width, about the origin, is

$$M^{(1)}(t) = \int_{-b}^{-a} y\{p^-(0,y,t) - p^+(0,y,t)\}\,dy \tag{53}$$

$$= \mathrm{Re}_j \frac{2\rho\sigma j\, e^{-j\sigma t}}{K^2} \int_{-b}^{-a} \frac{(1-Ky)}{((y^2-a^2)(b^2-y^2))^{\frac{1}{2}}} \{B - Cy^2 + F(a,b,y)\}\,dy. \tag{54}$$

It is well known (Ogilvie 1963; Haskind 1948) that it is possible to predict the time-averaged second-order forces and moments acting on an oscillating body in waves purely from a knowledge of the first-order potential Φ. Thus, the total force on the plate is given by integrating the exact expression for the pressure, namely,

$$p(x,y,t) = -\rho \frac{\partial \Phi_{\mathrm{ex.}}}{\partial t} - \rho g y - \tfrac{1}{2}|\nabla \Phi_{\mathrm{ex.}}|^2,$$

where $\Phi_{\mathrm{ex.}}$ is the exact velocity potential.

The notation

$$f^* = \frac{\sigma}{2\pi} \int_t^{(2\pi/\sigma)+t} f(t)\,dt$$

is used to denote time-averaged or mean quantities. It may be shown (Haskind 1959, p. 780) that the mean second-order horizontal force and moment per unit width are given by

$$X^{(2)*} = \tfrac{1}{4}\rho \int_{-b}^{-a} (|\nabla\phi^+|^2 - |\nabla\phi^-|^2)\,dy \tag{55}$$

and

$$M^{(2)*} = \tfrac{1}{4}\rho \int_{-b}^{-a} y(|\nabla\phi^+|^2 - |\nabla\phi^-|^2)\,dy, \tag{56}$$

where $\Phi(x,y,t)$, the first-order velocity potential, satisfies

$$\Phi(x,y,t) = \mathrm{Re}_j\{\phi\, e^{-j\sigma t}\}.$$

Now,

$$\phi^{\pm}(0,y) = e^{Ky}\{-jga_0/\sigma \pm \gamma(-K,y)\}, \quad y \text{ on } L,$$

and it may be shown that

$$X^{(2)*} = -\rho\sigma a_0 \,\mathrm{Im}_j \int_{-b}^{-a} e^{Ky} \frac{d}{dy}\{e^{Ky}\gamma(-K,y)\}\,dy$$

$$= -\tfrac{1}{2}\rho\sigma a_0 \,\mathrm{Im}_j\, \gamma(K), \tag{57}$$

after integration by parts twice.

Similarly,

$$M^{(2)*} = -\rho\sigma a_0 \operatorname{Im}_j \int_{-b}^{-a} y\, e^{Ky} \frac{d}{dy}\{e^{Ky}\gamma(-K,y)\}\,dy$$
$$= -\frac{\rho\sigma a_0}{4K}\operatorname{Im}_j\{\gamma(K)+2K\gamma_K(K)\}, \tag{58}$$

where $$\gamma_K(K) = \int_{-b}^{-a} \frac{u\,e^{Ku}}{((u^2-a^2)(b^2-u^2))^{\frac{1}{2}}}\{B-Cu^2+F(a,b,u)\}\,du. \tag{59}$$

Because of the presence of singularities in the velocity at the ends of the plate, and the corresponding negatively infinite pressures there, it may be shown that a mean second-order vertical force acts on the plate. See, for example, Sedov (1965), from which it follows that this force is given by

$$F = Y^{(2)*}(-a)+Y^{(2)*}(-b), \tag{60}$$

where $$Y^{(2)*}(-a) = \frac{-\rho}{4}\int_{C_a}\left|\frac{dw}{dz}\right|^2 dz \tag{61}$$

and the modulus sign refers to the complex number j.

Here C_a is a small contour surrounding the point $z = ia$, which is allowed to shrink to zero finally.

Similarly, $$Y^{(2)*}(-b) = \frac{-\rho}{4}\int_{C_b}\left|\frac{dw}{dz}\right|^2 dz. \tag{62}$$

Now from (21) and (22) we have, near $z = -ia$,

$$\frac{dw}{dz} = \frac{1}{(z+ia)^{\frac{1}{2}}\{-2ia(b^2-a^2)\}^{\frac{1}{2}}}$$
$$\times\{B-Ca^2+F(a,b,a)\}+\text{terms regular near } z = -ia, \tag{63}$$

with a similar behaviour near $z = -ib$, so that

$$F = \frac{\pi\rho}{4(b^2-a^2)}\left\{\frac{1}{a}|B-Ca^2+F(a,b,a)|^2-\frac{1}{b}|B-Cb^2+F(a,b,b)|^2\right\}, \tag{64}$$

being the contributions from the poles at $z = -ia$, $-ib$ as the contours shrink to zero.

6. Diffraction of waves by a fixed plate

The general expressions derived above permit the forces and moments acting on a submerged vertical plate, which is making small oscillations in a train of incident waves of the same frequency, to be determined by computing certain algebraic combinations of integrals.

In this paper, computations are restricted to a plate held fixed in a train of incident waves, in which case the results simplify considerably.

Thus the amplitude of roll θ_0 is zero, so that

$$f(y) = F(a,b,y) = \delta(K) = \Delta_{2i} \equiv 0,$$

whilst $$\Delta_3 = \frac{-ga_0}{\sigma}a_1(-K). \tag{65}$$

It also follows from (23) that, if we write

$$d^2 = B/C, \tag{66}$$

then

$$\int_a^b \frac{e^{Ku}(d^2-u^2)}{((u^2-a^2)(b^2-u^2))^{\frac{1}{2}}}\,du = 0, \tag{67}$$

which defines d^2.

The expressions for α, β, and γ simplify also and it is convenient to write

$$(\alpha', \beta', \gamma', \gamma'_K) = (1/C)(\alpha, \beta, \gamma, \gamma_K), \tag{68}$$

where

$$\gamma'(K) = \int_a^b \frac{e^{-Ku}(d^2-u^2)}{((u^2-a^2)(b^2-u^2))^{\frac{1}{2}}} \tag{69}$$

and α', β', γ'_K are defined similarly.

Then, from (37),

$$C = \frac{jga_0}{\sigma}\,\frac{1}{\gamma' - j(\alpha'-\beta')}. \tag{70}$$

For the fixed plate it is convenient to define a reflexion coefficient R as the ratio of the amplitude of the reflected wave to the amplitude of the incident wave, at infinity, and a transmission coefficient T as the ratio of the amplitude of the transmitted wave to the amplitude of the incident wave, at infinity.

Thus

$$R = \frac{|\gamma'(K)|}{(\gamma'^2+(\alpha'-\beta')^2)^{\frac{1}{2}}}, \tag{71}$$

whilst

$$T = \frac{|\alpha'-\beta'|}{(\gamma'^2+(\alpha'-\beta')^2)^{\frac{1}{2}}}, \tag{72}$$

where (4), (38), (39), (46) and (70) have been used.

Equation (52) for the oscillatory first-order horizontal force may be integrated in terms of tabulated functions when $F(a, b, u) = 0$.

Thus,

$$X^{(1)}(t) = \frac{-2\rho\sigma a_0}{Kb}\,\frac{\{d^2K(k) - b^2E(k)\}}{(\gamma'^2+(\alpha'-\beta')^2)^{\frac{1}{2}}}\cos(\sigma t - \epsilon) \tag{73}$$

where $k = (1-\mu^2)^{\frac{1}{2}}$, $\mu = a/b$, $K(k)$, $E(k)$ are the complete elliptic integrals of the first and second kind, respectively, and

$$\tan\epsilon = \frac{\alpha'-\beta'}{\gamma'}.$$

In a similar manner, from (54),

$$M^{(1)}(t) = \frac{2\rho\sigma a_0}{K^2b}\left\{d^2K(k) - b^2E(k) + \frac{\pi Kb}{4}(2d^2-a^2-b^2)\right\}\frac{\cos(\sigma t-\epsilon)}{(\gamma'^2+(\alpha'-\beta')^2)^{\frac{1}{2}}}. \tag{74}$$

It is useful to determine the centre of pressure of the first-order force as a fraction of the plate length measured from the top edge downwards. Thus, we define the centre of pressure $c_p^{(1)}$ by the equation,

$$\begin{aligned} c_p^{(1)} &\equiv -\left(\frac{M^{(1)}(t)}{X^{(1)}(t)} + a\right)\frac{1}{(b-a)} \\ &= \frac{1}{K(b-a)}\left\{1 - Ka + \frac{\pi Kb(2d^2-a^2-b^2)}{4\{d^2K(k)-b^2E(k)\}}\right\}. \end{aligned} \tag{75}$$

From (57) the second-order time-averaged horizontal force on the plate is simply

$$X^{(2)*} = -\tfrac{1}{2}\rho\sigma a_0 \gamma' \operatorname{Im}_j \left\{\frac{jga_0/\sigma}{\gamma' - j(\alpha' - \beta')}\right\}$$
$$= -\tfrac{1}{2}\rho g a_0^2 R^2. \qquad (76)$$

The second-order time-averaged moment becomes

$$M^{(2)*} = -\tfrac{1}{4}\rho g a_0^2 \frac{\gamma'}{\gamma'^2 + (\alpha' - \beta')}\left\{\frac{\gamma'}{K} + 2\gamma'_K\right\}, \qquad (77)$$

and the corresponding centre of pressure is

$$c_p^{(2)} = -\left(\frac{M^{(2)*}}{X^{(2)*}} + a\right)\frac{1}{b-a}$$
$$= -\frac{1}{K(b-a)}\left\{\tfrac{1}{2} + Ka + \frac{K\gamma'_K}{\gamma'}\right\}. \qquad (78)$$

Finally, the second-order mean vertical force given by (64), simplifies to

$$F = \frac{\pi\rho g a_0}{4K\{\gamma'^2 + (\alpha' - \beta')^2\}}\left\{\frac{(d^2-a^2)^2}{a} - \frac{(d^2-b^2)^2}{b}\right\}\frac{1}{(b^2-a^2)}. \qquad (79)$$

7. The limiting case $a = 0$

It is of interest to examine the behaviour of the various expressions for R, T, etc., as the top edge of the plate approaches the free surface. The reason for this is twofold. First, it provides a check on the expressions, which may be compared to those derived by Ursell (1947) and Haskind (1948, 1959) for the problem of diffraction of waves by a vertical surface-piercing plate. Secondly, one might anticipate some non-uniformity in the limiting process, since the behaviour of the fluid near the top edge of the plate, even when it is only slightly submerged, differs markedly from the behaviour of the fluid at the intersection of the surface-piercing plate and the free surface. Thus, the velocity is unbounded near either edge of the plate, and this is true however small the gap between the top of the plate and the free surface. On the other hand, the velocity of the fluid in the case of the surface-piercing plate is bounded at the intersection of the plate and free surface. This fundamental difference ought to be exhibited in the mathematical behaviour of the solution as $a \to 0$.

It is not difficult to show that this is indeed the case. Thus, as $\mu (= a/b) \to 0$, we find that

$$d^2 \to b^2[1 + \tfrac{1}{2}\pi(I_1(Kb) + L_1(Kb))]/\log(4/\mu),$$
$$\gamma'(K) \to b\pi I_1(Kb) + O((\log\mu)^{-1}),$$
$$\alpha'(K) \to \pi d^2 + O(\mu^2),$$
$$\beta'(K) \to -bK_1(Kb) + O((\log\mu)^{-1}),$$
$$\gamma'_K(K) \to b^2\{S_0(Kb) - S_1(Kb)\},$$

where

$$S_0(z) = \tfrac{1}{2}\pi[I_0(z) - L_0(z)] = \int_0^1 (1-t^2)^{-\frac{1}{2}} e^{-zt}\,dt,$$
$$S_1(z) = \frac{\pi}{2z}[I_1(z) - L_1(z)] = \int_0^1 (1-t^2)^{\frac{1}{2}} e^{-zt}\,dt,$$

I_0, I_1 and K_0, K_1 are modified Bessel functions of the first and second kind respectively, and L_0, L_1 are modified Struve functions (Watson 1940, p. 329).

We thus have the following limiting values for R and T:

$$R = \frac{\pi I_1(Kb)}{(\pi^2 I_1^2(Kb) + K_1^2(Kb))^{\frac{1}{2}}},$$

$$T = \frac{K_1(Kb)}{(\pi^2 I_1^2(Kb) + K_1^2(Kb))^{\frac{1}{2}}},$$

which agree with the values given by Ursell (1947).

Now, as $\mu \to 0$, $E(k) \to 1$, $K(k) \to \log(4/\pi)$, so that from (73), (74) and (75), for $\mu = 0$,

$$X^{(1)}(t) = \frac{-2\rho g a_0 b S_1(Kb)\cos(\sigma t - \epsilon)}{(\pi^2 I_1^2(Kb) + K_1^2(Kb))^{\frac{1}{2}}},$$

$$M^{(1)}(t) = \frac{2\rho g a_0 b\{S_1(Kb) - \frac{1}{4}\pi\}\cos(\sigma t - \epsilon)}{K(\pi^2 I_1^2(Kb) + K_1^2(Kb))^{\frac{1}{2}}},$$

and
$$c_p^{(1)} = \frac{1}{Kb}\left(1 - \frac{\pi}{4S_1(Kb)}\right),$$

where
$$\tan\epsilon = K_1(Kb)/\pi I_1(Kb).$$

It is easily seen from (76) that for $\mu = 0$,

$$X^{(2)*} = -\tfrac{1}{2}\rho g a_0^2 \frac{\pi^2 I_1^2(Kb)}{\pi^2 I_1^2(Kb) + K_1^2(Kb)} = -\tfrac{1}{2}\rho g a_0^2 R^2.$$

Also, from (77) and (79),

$$M^{(2)*} = -\tfrac{1}{2}\rho g a_0^2 b\left\{\frac{\pi I_1(Kb)}{2Kb} + S_0(Kb) - S_1(Kb)\right\}\frac{\pi I_1(Kb)}{\pi^2 I_1^2(Kb) + K_1^2(Kb)}$$

and
$$c_p^{(2)} = -\left\{\frac{1}{2Kb} + \frac{S_0(Kb) - S_1(Kb)}{\pi I_1(Kb)}\right\}.$$

These limiting values of the first- and second-order forces and moments agree with those given by Haskind (1959).

Finally, the vertical suction force increases without bound as $\mu \to 0$ and does not attain the negative value of the suction force on a surface-piercing barrier. This value is given by Haskind (1959) as

$$F = \frac{\pi\rho g a_0^2}{4Kb\{\pi^2 I_1^2(Kb) + K_1^2(Kb)\}},$$

which is the limiting value of (80) if the second term were zero. In fact this term is not zero, but dominates the expression for small μ, so that

$$F \to \frac{\pi\rho g a_0^2\{1 + KbS_1(-Kb)\}^2}{4Kb\{\pi^2 I_1^2(Kb) + K_1^2(Kb)\}\mu(\log(4/\mu))^2}$$

as $\mu \to 0$.

8. Discussion of results

In figure 2, curves are shown for the reflexion and transmission coefficients, R and T, as functions of $Kb\ (=\sigma^2 b/g)$ for different values of $\mu\ (=a/b)$. The limiting case $\mu = 0$, when the plate intersects the free surface, is shown for comparison. In contrast to this case, for each finite μ, R increases with increasing Kb until a maximum is reached and then decreases to zero for further increase in Kb. The

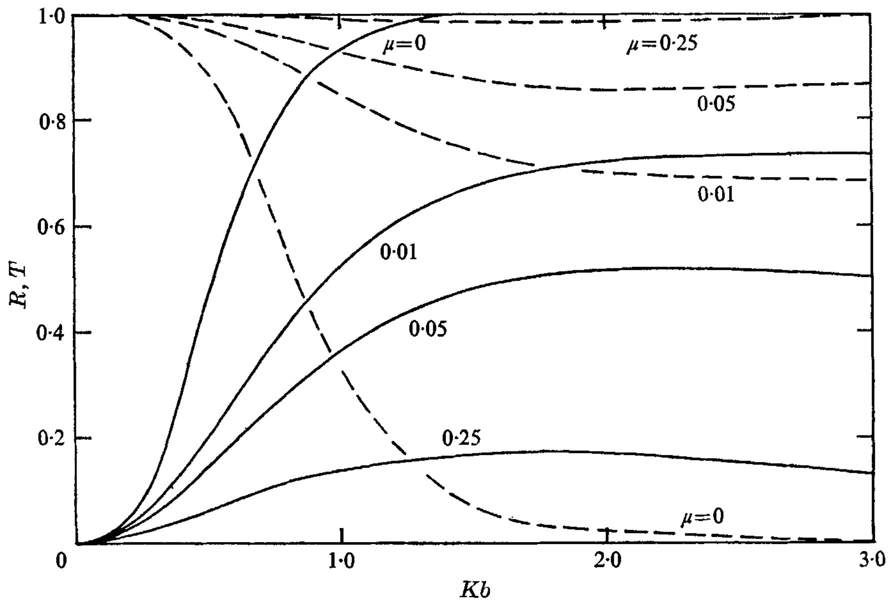

FIGURE 2. Reflexion coefficient R and transmission coefficient T vs. Kb for various μ. ———, R; – – – –, T.

physical explanation of the maximum in R is clear. For small values of Kb corresponding to an incident wave whose wavelength is large compared to the plate dimensions, the amount of reflexion is small, since the wave does not 'sense' the presence of the plate. As Kb increases, the incident wavelength and the plate dimension become comparable, and R increases. At the other extreme, for large Kb, the incident waves are short and the wave energy is confined to a thin surface layer, and transmission through the gap above the barrier occurs. Thus for each finite value of μ, $R \to 0$ as $Kb \to \infty$. Hence for some intermediate value of Kb, R must have a maximum.

It is clear from the curves that a submerged plate is a poor reflector of wave energy. For example, for $\mu = 0{\cdot}25$, R attains its maximum of about 0·18 when $Kb \doteqdot 1{\cdot}7$ indicating that less than 4 % of the wave energy is reflected for all wavelengths by a plate whose upper edge is submerged to one third of the total length of the plate. Even for $\mu = 0{\cdot}01$, so that there is only a small gap between the top of the plate and the free surface compared to the length of the plate, T is never less than 0·68, which means that at least 44 % of the wave energy is transmitted through the gap, for all wavelengths.

Figure 3 shows the amplitude of the first-order horizontal oscillatory force on the plate as a function of Kb for various μ including $\mu = 0$. It is noticeable how the sharp peak which occurs for $\mu = 0$ is reduced for finite μ. For instance, at $\mu = 0{\cdot}01$ the maximum value of the force is only about 60 % of its value at $\mu = 0$.

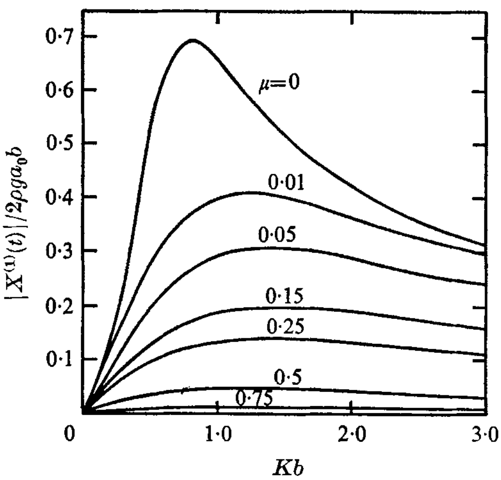

FIGURE 3. Amplitude of first-order horizontal force per unit width *vs.* Kb for various μ.

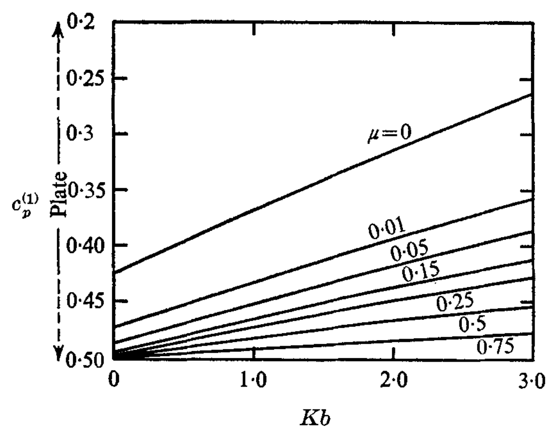

FIGURE 4. Centre of pressure of first-order oscillatory horizontal force $c_p^{(1)}$ *vs.* Kb for various μ.

In figure 4 the ordinate measures the position of the point of action, or centre of pressure, of the first-order oscillatory horizontal force, as a fraction of the plate length. For small values of Kb, the curves of $c_p^{(1)}$, for various μ, are indistinguishable from straight lines, although a slight curving towards the Kb axis is discernible for the larger values of Kb. This bending must take place, since for large Kb corresponding to short waves, the pressure will decay rapidly along the plate, and the force will act at a point very close to the top edge. Thus as $Kb \to \infty$, the curves tend asymptotically to zero. This also follows from a consideration of (75) for large Kb. The values of $c_p^{(1)}$ at $Kb = 0$ were obtained analytically from (75). It may be shown that, as $Kb \to 0$,

$$c_p^{(1)} \to \frac{1}{(1-\mu)}\left[\frac{K(k)}{3\pi}\left\{\frac{4(1-\frac{1}{2}k^2)P(k)-(1-k^2)-3P^2(k)}{1-\frac{1}{2}k^2-P(k)}\right\}-\mu\right],$$

where $$P(k) = E(k)/K(k) \quad (\mu^2 = 1-k^2),$$

which reduces to $4/3\pi$ for $\mu = 0$, and, after some algebra, to $\frac{1}{2}$ for $\mu = 1$.

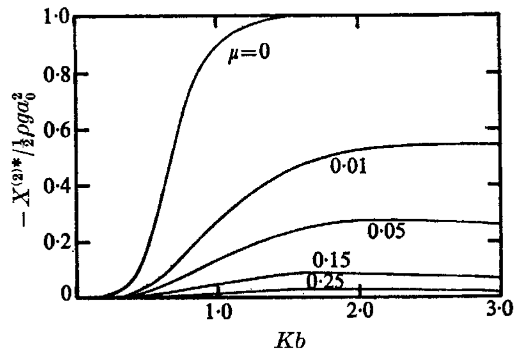

FIGURE 5. Mean second-order horizontal force *vs.* Kb for various μ.

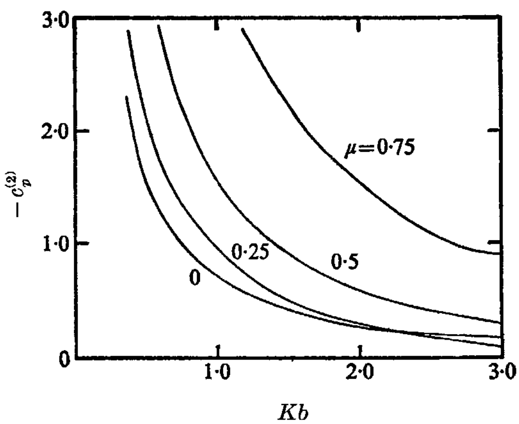

FIGURE 6. Point of action of mean second-order horizontal force *vs.* Kb for various μ.

It appears from the curves that for all Kb, $c_p^{(1)} \to \frac{1}{2}$ as $\mu \to 1$, indicating that for a short plate, the force is more uniformly distributed over the length than for a long plate. The fact that the centre of pressure lies on the upper half of the plate is consistent with the decay of gravity waves with depth.

Figure 5 shows the variation in the average second-order horizontal force as a function of Kb for various μ. Once again it is remarkable the effect a small gap above the plate makes to the force on the plate. The point of action of this force (figure 6) fluctuates markedly for different Kb and always lies above the top edge of the submerged plate, since $c_p^{(2)}$ is negative. However, the centre of pressure of the *total* force is dominated by the first-order force which we have seen always acts through a point on the plate.

Finally, figure 7 indicates the variation of the average second-order vertical force on the plate arising from the square-root singularities in velocity near the edges of the plate, and the correspondingly unbounded negative pressures there. As μ tends to zero, the force increases without bound for finite Kb and does not tend to the vertical suction force on a surface-piercing plate as given by Haskind,

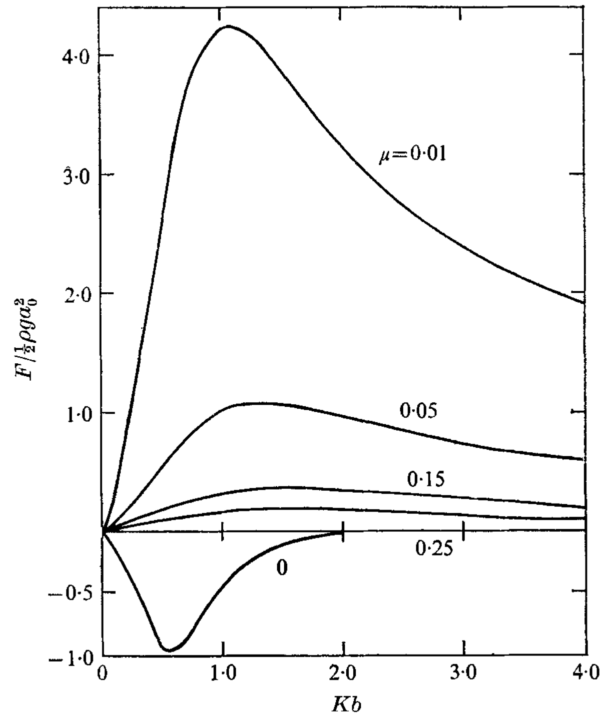

FIGURE 7. Mean second-order vertical force *vs.* Kb for various μ.

and indicated in the curve for $\mu = 0$. As long as there is the smallest gap between the top of the plate and the free surface, there will be a suction force arising from the singularity in velocity there, but when the plate intersects the surface the contribution to the forces arises solely from the lower edge of the plate and is directed down into the fluid, as illustrated in figure 7.

9. Conclusion

Results are obtained for the diffraction of water waves by a submerged thin vertical plate. In particular, computations are made of the reflexion and transmission coefficients and the first- and second-order forces and their points of application, when a wave of prescribed amplitude and phase is incident upon the plate. With the exception of the mean second-order vertical force on the plate, the results agree with those obtained by Ursell (1947) and Haskind (1959) for the special case $\mu = 0$ when the plate intersects the free surface.

Perhaps the most interesting feature of the results is their behaviour for small μ corresponding to a narrow gap between the top edge of the plate and the free surface. Thus the reluctance of the curves for small μ to approach the limiting curves $\mu = 0$ stems mathematically from the non-uniform behaviour of the solution as $\mu \to 0$. It has been shown that the solution for small μ cannot be expanded in a power series in μ about $\mu = 0$, but that the solution differs from the limiting case by a term of order $(\log (4/\mu))^{-1}$ as $\mu \to 0$. As already mentioned, this singular perturbation reflects the basic difference in the structure of the flow for $\mu = 0$ and μ finite. Thus, for $\mu = 0$ the velocity is bounded near the top edge of the plate, which is now in the surface. On the other hand, for each finite μ, however small, the velocity of the flow increases without bound for points arbitrarily close to the top edge of the plate.

In this connexion might be mentioned some recent work by Tuck (1970), who uses an approximate method based on the technique of matched asymptotic expansions to estimate the transmission coefficient for surface waves incident upon a vertical wall containing a deeply submerged narrow slit. He finds that a high transmission coefficient is possible for long waves passing through the submerged slit, which compares with the high transmission coefficients obtained in the present problem when the plate is close to the surface. It is not clear from Tuck's approximate solution why this should be so, and it is not possible to compare with the limiting cases solved by Dean (1945) or Ursell (1947) without violating the assumptions of this theory. It is possible that the exact solution, which may be obtained by the methods used in this paper, may provide the explanation.

The curves for very small μ must be treated with reservation since the linear theory is clearly not valid if the depth of submergence is comparable to the amplitude of the incident wave. Also, the infinite velocities predicted by the present linearized treatment will not occur in practice, of course. The behaviour of the flow near the plate will be influenced greatly by viscosity, which has been ignored in the present model, and vortices will be generated at the edges of the plate due to the oscillatory flow past the sharp edges.

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